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Modeling the Continental-Scale Dynamics of the Coupled Land-Surface and Atmospheric Water Balances with a Stochastic Differential Equation

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Abstract. Using a stochastic differential equation (SDE) approach, we examine the dynamics of the continental-scale water balance in the central United States. White-noise and colored-noise versions of the model are developed based on analysis of atmospheric data from the National Center for Environmental Prediction (NCEP) re-analysis project and long-term records of precipitation and soil moisture. Improved correspondence to observations is achieved by inclusion of terms in the SDE representing both hydrodynamic and thermodynamic soil-moisture feedbacks. We show that temporally correlated (*i.e.*, colored) atmospheric moisture flux removes the bimodality of the soil moisture probability distribution and that our colored-noise formulation successfully captures the water balance dynamics in the study region despite the absence of multiple soil moisture states. Specifically, the improved model reproduces both the autocorrelation structure and the inter-annual variability in the observations.

1. Introduction

Since the seminal work of Charney (Charney et al. 1977), land-surface feedbacks have increasingly been recognized as a significant source of atmospheric variability. Soil moisture is perhaps the most important mediator of these feedbacks. In this paper, we are specifically interested in the role soil moisture plays in the variability and persistence of precipitation at large spatial and temporal scales. The majority of previous modeling studies at this scale have involved the use of general circulation models (GCMs). Most GCM experiments are of the sensitivity variety in which surface conditions are altered in some significant way compared to a set of control conditions. While these studies are useful for understanding the physical processes involved, they generally are not suitable for fully quantifying the effect of the given feedback mechanism on the variability and persistence of precipitation at the seasonal to interannual scale. This is due in part to the necessary length of model runs and in part to questions as to the ability of GCMs to reproduce accurately variability at long time scales (Mearns et al 1990).

In contrast to GCMs, modeling methodologies are available which trade physical realism and detail for analytical tractability and computational ease (*e.g.*, Brubaker and Entekhabi 1995). At the very end of this side of the climate-modeling spectrum are lumped-parameter, continental-scale water balance models. In the present work, we employ one such model that is based

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on the work of Rodriguez-Iturbe et al. (1991) and others (Entekhabi et al. 1992, Wang et al. 1997). Because the model is first formulated as a single ordinary differential equation, issues of variability and persistence can be addressed in an analytical and probabilistic manner by reformulating the ordinary differential equation as a stochastic differential equation (SDE).

2. Deterministic Differential Equation Governing the Water Balance

We begin with the same ordinary differential equation governing the large-scale water balance used by Rodriguez-Iturbe et al. (1991):

$$nZ_r \frac{ds}{dt} = P(s)\phi(s) - E(s) \quad (1)$$

where

- s relative soil saturation (dimensionless);
- n soil porosity (dimensionless);
- Z_r hydrologically active depth of soil (L);
- $P(s)$ rainfall rate (L/T);
- $\phi(s)$ infiltration function (dimensionless);
- $E(s)$ evaporation rate (L/T).

The evaporation and infiltration functions are (Rodriguez-Iturbe et al):

$$E(s) = E_p s^c \quad (2)$$

$$\phi(s) = 1 - \varepsilon s^r \quad (3)$$

where

- E_p potential evaporation rate (L/T);
- c, ε, r nonnegative constants.

In the remainder of this paper it will be assumed that ε always takes on a value of one, ensuring that the runoff ratio is unity for completely saturated soils (*i.e.*, s=1).

Where we differ with Rodriguez-Iturbe et al. is in the formulation of the equation governing the rainfall rate. Whereas they assume that the portion of precipitation originating from atmospheric moisture that is advected to the region is constant, we assume that total precipitation is proportional to the average atmospheric moisture flux over the region. Following the streamtube analogy (*e.g.*, Entekhabi et al. 1992), we write

$$P(s) = h(s)(\mathrm{A} + \frac{E - P}{2}) \quad (4)$$

where

- A moisture flux per unit area advected to the region, (L/T)
- $h(.)$ fraction of moisture flux which falls as precipitation.

Solving for $P(s)$ gives

$$P(s) = \frac{h(s)}{1 + \frac{1}{2} h(s)} (\mathrm{A} + \tfrac{1}{2} E) \quad (5)$$

Because the energy balance at the surface controls the amount of atmospheric convection, and the energy balance is partially controlled by soil moisture through the Bowen ratio, we take h to be a function of soil moisture. For the study region, A~5 m/yr, P~0.9 m/yr and P-E~0.2 m/yr. Therefore, h~0.2 and $h \cong h/(1+ \frac{1}{2} h)$. Out of a desire for simplicity and flexibility, rather than from any theoretical basis, we use a power law relationship for $h(s)$, as was done for $E(s)$ and $\phi(s)$. Our equation for the precipitation rate as a function of soil moisture is thus

$$P(s) = (a + ks^d)(A + \tfrac{1}{2} E_p s^c) \quad (6)$$

where a, k and d are nonnegative constants (assuming a net positive feedback). Notice that s appears in two terms of (6). These terms represent *thermodynamic* feedbacks and direct *precipitation recycling*, respectively. Setting k or d equal to zero will remove the thermodynamic term, leaving only precipitation recycling as a feedback. We substitute (2), (3) and (6) into (1) and divide by nZ_r to arrive at a form of the differential equation for the water balance which involves only the time dimension:

$$ds/dt = (a + ks^d)(\alpha + \tfrac{1}{2} bs^c)(1 - \varepsilon s^r) - bs^c \quad (7)$$

where

$\alpha =$ A/nZ_r (T^{-1});
$b =$ E_p/nZ_r (T^{-1}).

3. White-Noise Formulation of the Stochastic Differential Equation

The transformation of (7) from a deterministic to a stochastic differential equation takes place with the treatment of α as a stochastic variable. α is a function of the input moisture flux and therefore can be viewed as the driver of the system (*e.g.*, Rodriguez-Iturbe et al. 1991). However, our formulation of the precipitation function has resulted in an α which is proportional to the input moisture flux (as opposed to the α of Rodriguez-Iturbe et al., which is inversely proportional.) We can view α as the sum of its mean plus a stochastic noise term:

$$\alpha = \overline{\alpha} + \sigma\gamma \quad (8)$$

where

$\overline{\alpha}$ the mean of α;
σ the standard deviation of α;
γ stochastic noise.

In addition, $E[\gamma] = 0$ and $Var[\gamma] = 1$. It is also typical to assume that the noise in an SDE is Gaussian. Wang et al. (1997) show that the Gaussian assumption is unnecessary for solution of the Fokker-Planck equation. Non-Gaussian noise is particularly useful for applications, such as the present, in which the driver takes on only positive values. For the purposes of making the SDE solvable using Ito calculus, the noise is also typically assumed to be white.

Substituting (8) into (7) gives a SDE of the Langevin form (Gardiner, 1985):

$$ds_t \big/ dt = G(s_t) + \sigma g(s_t)\,\gamma_t \tag{9}$$

where

G(s_t) drift term;

$\sigma g(s_t)$ diffusion term; and

$$G(s) = (a + ks^d)(\overline{\alpha} + \tfrac{1}{2} bs^c)(1 - s^r) - bs^c \tag{10a}$$

$$g(s) = (a + ks^d)(1 - s^r) \tag{10b}$$

The use of the terms *drift* and *diffusion* date from the original application of stochastic calculus to the study of Brownian motion and other diffusion processes. The subscript *t* has been used in the usual sense to indicate the stochastic time-dependence of a quantity. As in (10), we will drop the subscript when time is not explicitly part of the equation.

If we assume γ is a white noise process:

$$E[\gamma_t \gamma_s] = \delta(t - s) \tag{11}$$

Because white noise does not trace a differentiable path in time, it is not integrable by the normal rules of deterministic calculus. The stochastic integral of white noise is defined as the Weiner process W_t such that

$$dW_t = \gamma_t dt \tag{12}$$

where dW_t the derivative of the Weiner process. Using (12), we can write (9) for the white-noise case as

$$ds_t = G(s_t) + \sigma g(s_t)\, dW_t \tag{13}$$

The time evolution of the probability distribution of *s* is governed by the Ito Fokker-Planck equation (Gardiner, 1985):

$$\frac{\partial}{\partial t} f(s_t, t) = -\frac{\partial}{\partial s}[G(s_t) f(s_t, t)] + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial s^2}[g^2(s_t) f(s_t, t)] \tag{14}$$

where *f(s,t)* is the probability distribution of *s* at time *t*. Under steady state conditions, the LHS of (21) equals zero, $f(s,t) = f_{Ss}(s)$ with the solution,

$$f_{S_s}(s) = C\, exp\left(-\frac{2}{\sigma^2} \int_s \frac{G(u)}{g^2(u)}\, du + 2\, ln\, g(s) \right) \tag{15}$$

where *C* is a normalization constant such that

$$\int_0^1 f_{S_s}(s) ds = 1 \tag{16}$$

An analytical solution exists only if *a* = 0. This also turns out to be a condition for *s* = 0 to be a reflecting boundary. With *a* = 0, the solution of (15) is,

$$f_{S_s}(s) = \frac{C}{k^2 s^{2d}(1-s^r)^2} exp\left\{ \frac{2}{\sigma^2} \sum_{i=1}^{5} \left(U_i s^{rV_i} \left[\frac{1}{1-s^r} + (1-V_i) \right] \sum_{n=0}^{\infty} \frac{s^{nr}}{V_i + n} \right) \right\} \quad (17a)$$

where

$$\mathbf{U}_{\mathrm{T}} = \left[\frac{\overline{\alpha}}{kr}, -\frac{\overline{\alpha}}{kr}, \frac{b}{2kr}, -\frac{b}{2kr}, \frac{b}{k^2 r} \right] \quad (17b)$$

$$\mathbf{V}_{\mathrm{T}} = \left[\frac{1-d}{r}, 1+\frac{1-d}{r}, \frac{c-d+1}{r}, 1+\frac{c-d+1}{r}, \frac{c-2d+1}{r} \right] \quad (17c)$$

4. Colored-Noise Formulation of the Stochastic Differential Equation

The general approach to adding temporal correlation to α is to treat γ, its noise component in (15), as being generated by the Ornstein-Uhlenbeck process (*e.g.*, Wang et al., 1997). The origin of the Ornstein-Uhlenbeck process is in Langevin's model of Brownian motion (Gardiner, 1985) in which the velocity (rather than the position) of the particle is the principal stochastic variable subject to a white-noise forcing. The result is that the velocity is no longer non-differentiable in time (as in earlier models) and thus possesses temporal correlation. Conceptualizing γ as velocity in the Ornstein-Uhlenbeck process, we can write an SDE for γ of the form

$$d\gamma_t = -\frac{1}{\tau}\gamma_t dt + \sqrt{D}\, dW_t \quad (18)$$

where

τ correlation time scale of γ;
D diffusion coefficient.

Gardiner (1985) shows that

$$E[\gamma_t] = \gamma_{t=0}\, e^{-t/\tau} \quad (19)$$

$$Var[\gamma_t] = \frac{D\tau}{2}[1-e^{-2t/\tau}] \quad (20)$$

$$E[\gamma_t \gamma_s] = \frac{D\tau}{2} e^{-|t-s|/\tau} \quad (21)$$

In order that (20) converges to 1 as $t \to \infty$ and that (21) equals 1 when $t=s$,

$$D = \frac{2}{\tau} \quad (22)$$

Solving for γ in (9) and substituting into (18) along with (22) yields

$$\frac{d}{dt}\left(\frac{1}{g(s)}\left[\frac{ds}{dt}-G(s)\right]\right)=-\frac{1}{\tau}\left(\frac{1}{g(s)}\left[\frac{ds}{dt}-G(s)\right]\right)+\sigma\sqrt{\frac{2}{\tau}}\frac{dW}{dt} \qquad (23)$$

Applying the chain rule of ordinary calculus to the LHS of (34) gives

$$\frac{d^2s}{dt^2}=\frac{1}{g(s)}\frac{\partial g(s)}{\partial s}\left(\frac{ds}{dt}\right)^2+\left[g(s)\frac{\partial}{\partial s}\left(\frac{G(s)}{g(s)}\right)-\frac{1}{\tau}\right]\frac{ds}{dt}+\frac{G(s)}{\tau}+\sqrt{\frac{2}{\tau}}\sigma g(s)\frac{dW}{dt} \qquad (24)$$

Discarding the squared and second-derivative terms,

$$ds=\frac{G(s)dt+\sqrt{2\tau}\,g(s)dW}{1-\tau g(s)\frac{d}{ds}\left(\frac{G(s)}{g(s)}\right)} \qquad (25)$$

We define new drift and diffusion terms,

$$\mathsf{G}(s)=\frac{G(s)}{1-\tau g(s)\frac{d}{ds}\left(\frac{G(s)}{g(s)}\right)} \qquad (26)$$

$$\mathsf{g}(s)=\frac{\sqrt{2\tau}\,g(s)}{1-\tau g(s)\frac{d}{ds}\left(\frac{G(s)}{g(s)}\right)} \qquad (27)$$

in order to write (25) in standard Langevin white-noise form:

$$ds_t=\mathsf{G}(s_t)dt+\mathsf{g}(s_t)dW_t \qquad (28)$$

As in the white-noise case, choosing a=0 allows for an analytical solution. This solution, which contains 37 series terms of the form in (17), is too long to reproduce in this paper.

5. Model Parameterization and Solutions for the Study Region

The study region is the graticule bounded by 32.5° N, 42.5° N, 87.5° W and 102.5° W. The boundaries of the region are overlain onto a map of mean annual precipitation in Figure 1. Long-term, large-scale data sets of wind speed, humidity, precipitation, potential evapotranspiration, runoff and soil moisture were used to develop a parameterization of the model for the study region. A daily version of results from the NCEP/NCAR re-analysis project (Kalnay et al. 1996) was the source of the wind speed and humidity data. These data are resolved vertically by 14 pressure levels and horizontal at 0.5 degrees latitude and longitude. The precipitation data were extracted from a monthly data set generated by the PRISM model (Daly et al. 1994) for the entire United States at a 10-km resolution. The two data sets overlap over a

37-year period (1958-1993). A long-term mean value of potential evaporation over the study region was derived from estimates made by Hobbins et al (2000) for the entire U.S. at a 10-km resolution based on modified version of the Penman-Monteith equation.

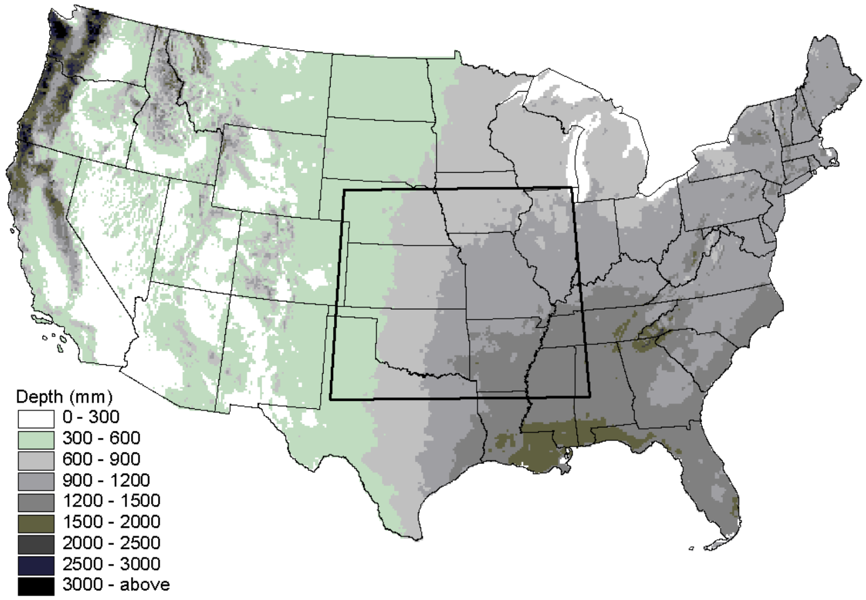

Figure 1: Study region and mean annual precipitation

Unfortunately, reliable soil moisture data at similar spatial and temporal scales do not exist. To our knowledge, the best set of long-term, large-scale observations of soil moisture in the U.S. is the "soil moisture climatology" developed by the Illinois Water Resources Survey (Hollinger and Isard 1994). We used data from 16 of the ISWR sites to develop monthly state-wide averages of relative soil saturation for the years 1983-1995. Data from a similar number of USGS stream gauges were used to estimate annual runoff over the same period. As shown in Table 1, the hydroclimatic characteristics of Illinois and the larger study region are fairly similar. Therefore, it should be reasonable to extrapolate values of the soil parameters *nZr*, *c* and *r*, as derived for Illinois, to the larger study region. The estimated values of these and the remaining parameters are given in Table 2. The exponents *c*, *r* and *d* have been rounded to whole and half-integer values to increase the analytical tractability of the solutions of the Fokker-Plank equation. The analysis of the precipitation data and the atmospheric moisture flux estimates support inclusion of the thermodynamic feedback factor.

Table 1: Hydroclimatic Characteristics of Data Locations

	LOCATION		
	Study Region	*Illinois*	*R5 Watershed*
Data Years	1958 – 1993	1983 – 1995	1967 – 1973
Area (sq. km)	1.86E+06	1.45E+05	0.1
Mean Annual:			
Precipitation, P (m)	0.91	1.04	0.77
Potential Evap., E_p (m)	1.5	1.22	1.8
Runoff, Q (m)	0.19	0.33	0.06
E_p/P	1.65	1.17	2.34
Q/P	0.21	0.32	0.08
Effective Soil Saturation	?	0.71	0.46

Table 2: Parameter values

Parameter	*Value*
Mean rate of advected moisture input, A (m/yr)	4.7
Standard Deviation of A, σ_A (m/yr)	2.7
Correlation time scale of A, τ (yrs)	0.0079
Potential evaporation rate, E_p (m/yr)	1.5
Storage capacity of active soil depth, nZ_r (m)	0.6
Evaporation function exponent, c	1.5
Runoff function exponent, r	4
Precipitation factor constant, a	0
Precipitation factor coefficient, k	0.29
Precipitation factor exponent, d	1

For the parameter values in Table 2, the analytical solutions to the white- and colored-noise formulations of the Fokker-Planck equation are compared in Figure 2. As does the model of Rodriguez-Iturbe et al, our model produces a bimodal distribution of soil moisture under a white-noise solution. However, in marked contrast to both our white-noise solution and the colored-noise solution of Wang et al., our colored-noise solution has a classical Gaussian shape. Clearly, white-noise and colored-noise processes are fundamentally different. Given that the atmospheric moisture flux is correlated in time, one hopes that the colored-noise model better represents the underlying dynamics.

6. Comparison of Model Results to Observations

For model validation, as with the parameterization, we would ideally like a multi-decadal time series of observational estimates of area-averaged soil moisture over the study region. In the absence of such a data set, we again turn to the Illinois data. The area and climate of Illinois are close enough to those of the larger study region to make comparison of the statistical properties of the model results and the observational data worthwhile.

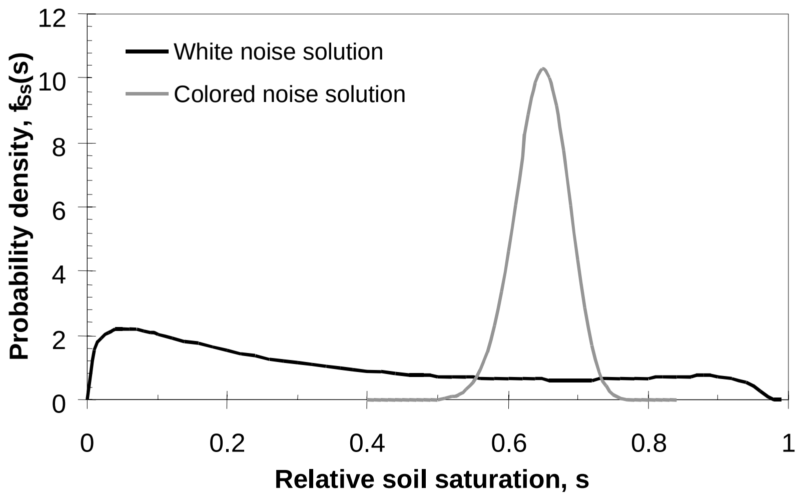

Figure 2. Comparison of white noise and colored noise solutions

The most basic statistics that we can compare are the expected value of $f_{Ss}(s)$ with the long-term mean of the Illinois data, which, as listed in Table 1, is 0.71. The expected value of the white-noise distribution shown in Figure 2 is 0.37. The expected value of the colored-noise distribution is 0.65. Given that the climate of the study region is only slightly drier than that of Illinois, the colored-noise formulation appears to produce the better estimate of mean soil moisture.

There are also enough observations to make reasonably accurate variance and auto-covariance calculations with the Illinois data. Based on the SDE model, the auto-covariance function for soil moisture can be derived from a temporally dependent solution of the Fokker-Plank equation. Because there is no general analytical solution (unlike the steady-state solution), we have resorted to a numerical solution, the details of which are beyond the scope of this paper. The variance of the temporal average, and the auto-covariance as a function of time, are plotted in Figure 3 for both the colored-noise and white-noise models. Also plotted are the variance and its 95% confidence limits for the annual-average soil moisture for the 13 complete years of the Illinois data. The white-noise formulation overestimates the variance of the annual-average soil moisture by an order of magnitude. The colored-noise formulation appears to produce less variability then found in the Illinois data. This is to be expected given that the Illinois data represent a smaller spatial scale and also likely contain significant measurement and sampling error.

The amount of persistence in the two models and the Illinois data can be compared directly with the autocorrelation plot of Figure 4. Autocorrelation coefficients for the Illinois data were calculated from seasonally standardized monthly averages. Given this fact and those noted above, it is expected that the Illinois coefficients would fall below the autocorrelation function for the study region. Nevertheless, the white-noise formulation appears to over-predict greatly the persistence of soil moisture.

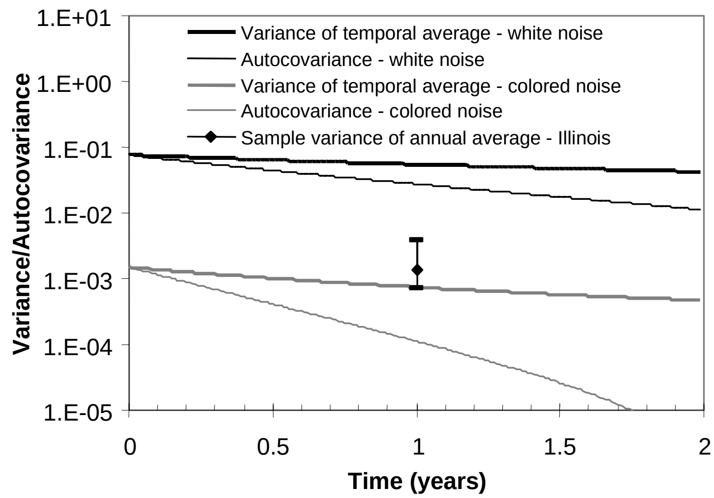

Figure 3. Autocovariance and variance of temporal-average soil moisture as calculated using colored and white noise

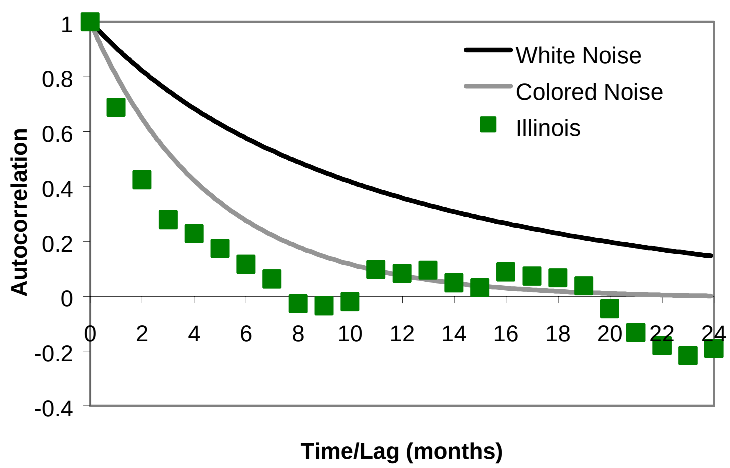

Figure 4. Comparison of white-noise and colored-noise autocorrelation functions with correlogram of standardized monthly soil moisture from Illinois

By way of (6) and probability distributions of A and s, an analytical form for the probability distribution of precipitation can be derived. However, because of the lack of seasonality in the model and the highly episodic nature of precipitation at any scale, this distribution has no observable counterpart. We have developed a quasi-analytical method for deriving the probability distributions of both temporally averaged precipitation and soil moisture. It is, however, less computationally efficient than generating frequency distributions from numerical simulation of the SDE. Accordingly, frequency distributions of annual precipitation were produced from 50,000-year simulations that were driven by atmospheric moisture flux generated using the first-order gamma autoregressive model of Fernandez and Salas (1990). Plotted on a normal probability scale in Figure 5 are modeled frequency distributions of annual average precipitation for the parameterization in Table 1 and a parameterization, which removes the thermodynamic feedback term. Also plotted are the 37 PRISM-based observational values that were used in the parameterization. The sample variance of the observations is 116 cm^2 as compared to 110 cm^2 and 43 cm^2 for the thermodynamic-feedback and precipitation-recycling-only cases, respectively. These results suggest not only that the colored-noise formulation of the model captures most of the inter-annual variability of precipitation but that thermodynamic feedbacks are a significant source of that variability.

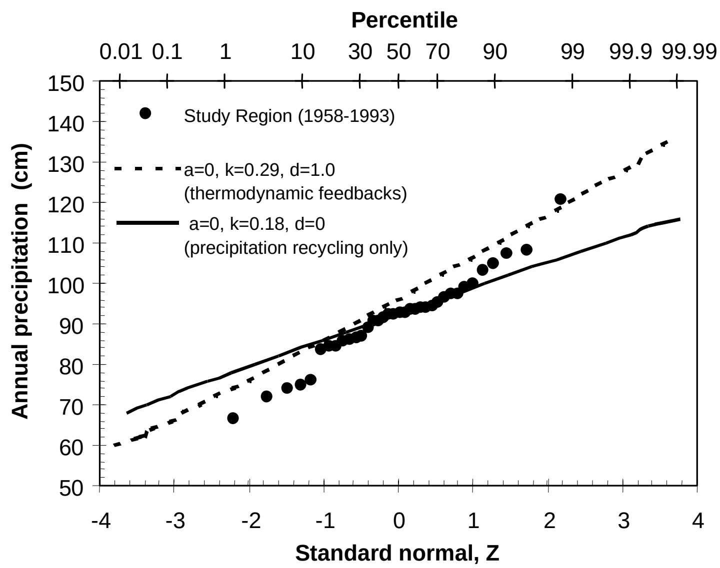

Figure 5. Modeled frequency distributions of annual precipitation compared to the observations used in the parameterization and observations from a longer time series.

7. Summary

Based on a lumped-parameter water balance model, we have developed a stochastic differential equation that describes the dynamics of continental-scale atmospheric moisture fluxes and soil moisture. Despite its simplicity, it reproduced quite well the amount of variability and persistence in large-scale, long-term observations of soil moisture and precipitation in the central United States. It was shown that a colored-noise formulation of the SDE is essential for agreement with the observations. While a white-noise formulation produces the very interesting and implication-ridden case of a bimodal/multi-state probability distribution for soil moisture, we believe it does not represent the actual system in the central United States. In contrast, the colored-noise formulation seems to capture the positive soil-moisture feedbacks that are likely a significant contributor to inter-annual precipitation variability in the study region.

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